GEOMETRIC INTERPRETATION OF THE INVARIANTS OF A SURFACE IN $\mathbb{R}^4$ VIA THE TANGENT INDICATRIX AND THE NORMAL CURVATURE ELLIPSE

GEORGI GANCHEV AND VELICHKA MILOUSHEVA

Abstract. At any point of a surface in the four-dimensional Euclidean space we consider the geometric configuration consisting of two figures: the tangent indicatrix, which is a conic in the tangent plane, and the normal curvature ellipse. We show that the basic geometric classes of surfaces in the four-dimensional Euclidean space, determined by conditions on their invariants, can be interpreted in terms of the properties of the two geometric figures. We give some non-trivial examples of surfaces from the classes in consideration.

1. Introduction

In this paper we deal with the theory of surfaces in the four-dimensional Euclidean space $\mathbb{R}^4$.

Let M^2 be a surface in $\mathbb{R}^4$ with tangent space T_pM^2 at any point $p \in M^2$. In [4] we introduced the linear map γ of Weingarten type at any T_pM^2 and sketched out the invariant theory of surfaces on the base of γ.

We show that the role of the map γ in the theory of surfaces in $\mathbb{R}^4$ is similar to that of the Weingarten map in the theory of surfaces in $\mathbb{R}^3$.

First, the map γ generates two invariant functions k and $\varkappa$, analogous to the Gauss curvature and the mean curvature in $\mathbb{R}^3$. Here again the sign of the function k is a geometric invariant and the sign of $\varkappa$ is invariant under the motions in $\mathbb{R}^4$. However, the sign of $\varkappa$ changes under symmetries with respect to a hyperplane in $\mathbb{R}^4$. The invariants k and $\varkappa$ divide the points of M^2 into four types: flat, elliptic, hyperbolic and parabolic points. In [4] we gave a constructive classification of the surfaces consisting of flat points, i.e. satisfying the condition $k = \varkappa = 0$. Everywhere, in the present considerations we exclude the points at which $k = \varkappa = 0$.

Further, the map γ generates the second fundamental form II at any point $p \in M^2$. The notions of a normal curvature of a tangent, conjugate and asymptotic tangents are introduced in the standard way by means of II. The asymptotic tangents are characterized by zero normal curvature.

The first fundamental form I and the second fundamental form II generate principal tangents and principal lines, as in $\mathbb{R}^3$. Here, the points at which any tangent is principal ("umbilical" points) are characterized by zero mean curvature vector, i.e. the surfaces consisting of "umbilical" points are exactly the minimal surfaces in $\mathbb{R}^4$. The principal normal curvatures ν' and ν'' arise in the standard way and the invariants k and $\varkappa$ satisfy the equalities

$$k = \nu'\nu''; \qquad \varkappa = \frac{\nu' + \nu''}{2}.$$

The indicatrix of Dupin at an arbitrary (non-flat) point of a surface in $\mathbb{R}^3$ is introduced by means of the second fundamental form. Here, using the second fundamental form II, we

2000 *Mathematics Subject Classification.* Primary 53A07, Secondary 53A10.

Key words and phrases. Surfaces in the four-dimensional Euclidean space, Weingarten map, tangent indicatrix, normal curvature ellipse.

introduce the indicatrix χ at any point $p \in M^2$ in the same way:

$$\chi : \nu' X^2 + \nu'' Y^2 = \varepsilon, \qquad \varepsilon = \pm 1.$$

Then the elliptic, hyperbolic and parabolic points of a surface M^2 are characterized in terms of the indicatrix χ as in $\mathbb{R}^3$. The conjugacy in terms of the second fundamental form coincides with the conjugacy with respect to the indicatrix χ.

In [4, 5] we proved that the surface M^2 under consideration is with flat normal connection if and only if $\varkappa = 0$. In Section 3 we prove that:

The surface M^2 is minimal if and only if the indicatrix χ is a circle.

The surface M^2 is with flat normal connection if and only if the indicatrix χ is a rectangular hyperbola (a Lorentz circle).

We also characterize the surfaces with flat normal connection in terms of the properties of the normal curvature ellipse.

In Section 4 we give examples of surfaces with $\varkappa = 0$.

in Section 5 we give examples of surfaces with $k = 0$.

2. An interpretation of the second fundamental form

Let $M^2 : z = z(u, v),\ (u, v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a 2-dimensional surface in $\mathbb{R}^4$. The tangent space T_pM^2 to M^2 at an arbitrary point $p = z(u, v)$ of M^2 is span$\{z_u, z_v\}$. We choose an orthonormal normal frame field $\{e_1, e_2\}$ of M^2 so that the quadruple $\{z_u, z_v, e_1, e_2\}$ is positive oriented in $\mathbb{R}^4$. Then the following derivative formulas hold:

$$\begin{array}{l}
\nabla'_{z_u} z_u = z_{uu} = \Gamma_{11}^1\, z_u + \Gamma_{11}^2\, z_v + c_{11}^1\, e_1 + c_{11}^2\, e_2,\\
\nabla'_{z_u} z_v = z_{uv} = \Gamma_{12}^1\, z_u + \Gamma_{12}^2\, z_v + c_{12}^1\, e_1 + c_{12}^2\, e_2,\\
\nabla'_{z_v} z_v = z_{vv} = \Gamma_{22}^1\, z_u + \Gamma_{22}^2\, z_v + c_{22}^1\, e_1 + c_{22}^2\, e_2,
\end{array}$$

where Γ_{ij}^k are the Christoffel's symbols and $c_{ij}^k,\ i, j, k = 1, 2$ are functions on M^2.

We use the standard denotations $E(u, v) = g(z_u, z_u),\ F(u, v) = g(z_u, z_v),\ G(u, v) = g(z_v, z_v)$ for the coefficients of the first fundamental form and set $W = \sqrt{EG - F^2}$. Denoting by σ the second fundamental tensor of M^2, we have

$$\begin{array}{l}
\sigma(z_u, z_u) = c_{11}^1\, e_1 + c_{11}^2\, e_2,\\
\sigma(z_u, z_v) = c_{12}^1\, e_1 + c_{12}^2\, e_2,\\
\sigma(z_v, z_v) = c_{22}^1\, e_1 + c_{22}^2\, e_2.
\end{array}$$

In [4] we introduced a geometrically determined linear map γ in the tangent space at any point of a surface M^2 and found invariants generated by this map.

We consider the functions

$$L = \frac{2}{W}\left| \begin{array}{cc} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{array} \right|, \quad M = \frac{1}{W}\left| \begin{array}{cc} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{array} \right|, \quad N = \frac{2}{W}\left| \begin{array}{cc} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{array} \right|.$$

Denoting

$$\gamma_1^1 = \frac{FM - GL}{EG - F^2}, \quad \gamma_1^2 = \frac{FL - EM}{EG - F^2}, \quad \gamma_2^1 = \frac{FN - GM}{EG - F^2}, \quad \gamma_2^2 = \frac{FM - EN}{EG - F^2},$$

we obtain the linear map

$$\gamma : T_pM^2 \to T_pM^2,$$

determined by the equalities

$$\gamma(z_u) = \gamma_1^1 z_u + \gamma_1^2 z_v,$$
$$\gamma(z_v) = \gamma_2^1 z_u + \gamma_2^2 z_v.$$

The linear map γ of Weingarten type at the point $p \in M^2$ is invariant with respect to changes of parameters on M^2 as well as to motions in $\mathbb{R}^4$. This implies that the functions

$$k = \frac{LN - M^2}{EG - F^2}, \qquad \varkappa = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

are invariants of the surface M^2.

The invariant $\varkappa$ is the curvature of the normal connection of the surface M^2 in $\mathbb{E}^4$.

The invariants k and $\varkappa$ divide the points of M^2 into four types [4]: flat, elliptic, parabolic and hyperbolic. The surfaces consisting of flat points satisfy the conditions

$$k(u,v) = 0, \quad \varkappa(u,v) = 0, \qquad (u,v) \in \mathcal{D},$$

or equivalently $L(u,v) = 0,\ M(u,v) = 0,\ N(u,v) = 0,\ (u,v) \in \mathcal{D}$. These surfaces are either planar surfaces (there exists a hyperplane $\mathbb{R}^3 \subset \mathbb{R}^4$ containing M^2) or developable ruled surfaces.

Further we consider surfaces free of flat points, i.e. $(L, M, N) \neq (0, 0, 0)$.

Let $X = \alpha z_u + \beta z_v,\ (\alpha, \beta) \neq (0,0)$ be a tangent vector at a point $p \in M^2$. The Weingarten map γ determines a second fundamental form of the surface M^2 at $p \in M^2$ as follows:

$$II(\alpha, \beta) = -g(\gamma(X), X) = L\alpha^2 + 2M\alpha\beta + N\beta^2, \quad \alpha, \beta \in \mathbb{R}.$$

As in the classical differential geometry of surfaces in $\mathbb{R}^3$ the second fundamental form II determines conjugate tangents at a point p of M^2.

Two tangents $g_1 : X = \alpha_1 z_u + \beta_1 z_v$ and $g_2 : X = \alpha_2 z_u + \beta_2 z_v$ are said to be *conjugate tangents* if $II(\alpha_1, \beta_1; \alpha_2, \beta_2) = 0$, i.e.

$$L\alpha_1\alpha_2 + M(\alpha_1\beta_2 + \alpha_2\beta_1) + N\beta_1\beta_2 = 0.$$

A tangent $g : X = \alpha z_u + \beta z_v$ is said to be *asymptotic* if it is self-conjugate, i.e. $L\alpha^2 + 2M\alpha\beta + N\beta^2 = 0$.

A tangent $g : X = \alpha z_u + \beta z_v$ is said to be *principal* if it is perpendicular to its conjugate. The equation for the principal tangents at a point $p \in M^2$ is

$$\begin{vmatrix} E & F \\ L & M \end{vmatrix} \alpha^2 + \begin{vmatrix} E & G \\ L & N \end{vmatrix} \alpha\beta + \begin{vmatrix} F & G \\ M & N \end{vmatrix} \beta^2 = 0.$$

A line $c : u = u(q),\ v = v(q);\ q \in J$ on M^2 is said to be a *principal line* (a *line of curvature*) if its tangent at any point is principal. The surface M^2 is parameterized with respect to the principal lines if and only if

$$F = 0, \qquad M = 0.$$

Let M^2 be parameterized with respect to the principal lines and denote the unit vector fields $x = \dfrac{z_u}{\sqrt{E}},\ y = \dfrac{z_v}{\sqrt{G}}$. The equality $M = 0$ implies that the normal vector fields $\sigma(x,x)$ and $\sigma(y,y)$ are collinear. We denote by b a unit normal vector field collinear with $\sigma(x,x)$ and $\sigma(y,y)$, and by l the unit normal vector field such that $\{x, y, b, l\}$ is a positive oriented orthonormal frame field of M^2 (the two vectors $\{b, l\}$ are determined up to a sign). Thus we

obtain a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point $p \in M^2$. With respect to the frame field $\{x, y, b, l\}$ we have the following formulas:

$$\begin{aligned} \sigma(x,x) &= \nu_1\, b; \\ \sigma(x,y) &= \lambda\, b + \mu\, l; \\ \sigma(y,y) &= \nu_2\, b, \end{aligned} \tag{2.1}$$

where $\nu_1, \nu_2, \lambda, \mu$ are invariant functions, whose signs depend on the pair $\{b, l\}$.

Hence the invariants k, $\varkappa$, and the Gauss curvature K of M^2 are expressed as follows:

$$k = -4\nu_1\,\nu_2\,\mu^2, \qquad \varkappa = (\nu_1 - \nu_2)\mu, \qquad K = \nu_1\,\nu_2 - (\lambda^2 + \mu^2). \tag{2.2}$$

The normal mean curvature vector field H of M^2 is $H = \dfrac{\sigma(x,x) + \sigma(y,y)}{2} = \dfrac{\nu_1 + \nu_2}{2}\, b$.

Let M^2 be a surface parameterized by principal tangents and $g : X = \alpha z_u + \beta z_v$ be an arbitrary tangent of M^2. We call the function $\nu_g = \dfrac{II(\alpha,\beta)}{I(\alpha,\beta)}$ the *normal curvature* of g. Obviously, a tangent g is asymptotic if and only if its normal curvature is zero.

The normal curvatures $\nu' = \dfrac{L}{E}$ and $\nu'' = \dfrac{N}{G}$ of the principal tangents are said to be *principal normal curvatures* of M^2. If g is an arbitrary tangent with normal curvature ν_g, and $\varphi = \angle(g, z_u)$, then the following Euler formula holds

$$\nu_g = \cos^2\varphi\,\nu' + \sin^2\varphi\,\nu''.$$

The invariants k and $\varkappa$ of M^2 are expressed by the principal normal curvatures ν' and ν'' as follows:

$$k = \nu'\nu''; \qquad \varkappa = \frac{\nu' + \nu''}{2}. \tag{2.3}$$

Hence, the invariants k and $\varkappa$ of M^2 play the same role in the differential geometry of surfaces in $\mathbb{R}^4$ as the Gaussian curvature and the mean curvature in the classical differential geometry of surfaces in $\mathbb{R}^3$.

As in the theory of surfaces in $\mathbb{R}^3$, we consider the indicatrix χ in the tangent space T_pM^2 at an arbitrary point p of M^2, defined by

$$\chi : \nu' X^2 + \nu'' Y^2 = \varepsilon, \qquad \varepsilon = \pm 1.$$

If p is an elliptic point ($k > 0$), then the indicatrix χ is an ellipse. The axes of χ are collinear with the principal directions at the point p, and the lengths of the axes are $\dfrac{2}{\sqrt{|\nu'|}}$ and $\dfrac{2}{\sqrt{|\nu''|}}$.

If p is a hyperbolic point ($k < 0$), then the indicatrix χ consists of two hyperbolas. For the sake of simplicity we say that χ is a hyperbola. The axes of χ are collinear with the principal directions, and the lengths of the axes are $\dfrac{2}{\sqrt{|\nu'|}}$ and $\dfrac{2}{\sqrt{|\nu''|}}$.

If p is a parabolic point ($k = 0$), then the indicatrix χ consists of two straight lines parallel to the principal direction with non-zero normal curvature.

The following statement holds good:

Proposition 2.1. *Two tangents g_1 and g_2 are conjugate tangents of M^2 if and only if g_1 and g_2 are conjugate with respect to the indicatrix χ.*

3. Classes of surfaces characterized in terms of the tangent indicatrix and the normal curvature ellipse

Each surface M^2 in $\mathbb{R}^4$ satisfies the following inequality:

$$\varkappa^2 - k \geq 0.$$

The minimal surfaces in $\mathbb{R}^4$ are characterized by

Proposition 3.1. [4] *Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. Then M^2 is minimal if and only if*

$$\varkappa^2 - k = 0.$$

The surfaces with flat normal connection are characterized by

Proposition 3.2. *Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. Then M^2 is a surface with flat normal connection if and only if*

$$\varkappa = 0.$$

We note that the condition $\varkappa = 0$ implies that $k < 0$ and the surface M^2 has two families of orthogonal asymptotic lines.

Now we shall characterize the minimal surfaces and the surfaces with flat normal connection in terms of the tangent indicatrix of the surface.

Proposition 3.3. *Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. Then M^2 is minimal if and only if at each point of M^2 the tangent indicatrix χ is a circle.*

Proof: Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. From equalities (2.3) it follows that

$$\varkappa^2 - k = \left(\frac{\nu' - \nu''}{2}\right)^2.$$

Obviously $\varkappa^2 - k = 0$ if and only if $\nu' = \nu''$. Applying Proposition 3.1, we get that M^2 is minimal if and only if χ is a circle. □

Proposition 3.4. *Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. Then M^2 is a surface of flat normal connection if and only if at each point of M^2 the tangent indicatrix χ is a rectangular hyperbola (a Lorentz circle).*

Proof: Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. From (2.3) it follows that $\varkappa = 0$ if and only if $\nu'' = -\nu'$.

If M^2 is a surface with flat normal connection, then $k < 0$, and hence χ is a hyperbola. From $\nu'' = -\nu'$ it follows that the semi-axes of χ are equal to $\dfrac{1}{\sqrt{|\nu'|}}$, i.e. χ is a rectangular hyperbola.

Conversely, if χ is a rectangular hyperbola, then $\nu'' = -\nu'$, which implies that M^2 is a surface with flat normal connection. □

The minimal surfaces and the surfaces with flat normal connection can also be characterized in terms of the ellipse of normal curvature.

Let us recall that the *ellipse of normal curvature* at a point p of a surface M^2 in $\mathbb{R}^4$ is the ellipse in the normal space at the point p given by $\{\sigma(x,x) : x \in T_pM^2,\ g(x,x) = 1\}$ [7, 8]. Let $\{x, y\}$ be an orthonormal base of the tangent space T_pM^2 at p. Then, for any $v = \cos\psi\, x + \sin\psi\, y$, we have

$$\sigma(v,v) = H + \cos 2\psi\, \frac{\sigma(x,x) - \sigma(y,y)}{2} + \sin 2\psi\, \sigma(x,y),$$

where $H = \dfrac{\sigma(x,x)+\sigma(y,y)}{2}$ is the mean curvature vector of M^2 at p. So, when v goes once around the unit tangent circle, the vector $\sigma(v,v)$ goes twice around the ellipse centered at H. The vectors $\dfrac{\sigma(x,x)-\sigma(y,y)}{2}$ and $\sigma(x,y)$ determine conjugate directions of the ellipse.

A surface M^2 in $\mathbb{R}^4$ is called *super-conformal* [3] if at any point of M^2 the ellipse of curvature is a circle. In [3] it is given an explicit construction of any simply connected super-conformal surface in $\mathbb{R}^4$ that is free of minimal and flat points.

Obviously, M^2 is minimal if and only if for each point $p \in M^2$ the ellipse of curvature is centered at p.

The minimal surfaces in $\mathbb{R}^4$ are divided into two subclasses:

- the subclass of minimal super-conformal surfaces, characterized by the condition that the ellipse of curvature is a circle;
- subclass of minimal surfaces of general type, characterized by the condition that the ellipse of curvature is not a circle.

In [5] it is proved that on any minimal surface M^2 the Gauss curvature K and the normal curvature $\varkappa$ satisfy the following inequality

$$K^2 - \varkappa^2 \geq 0.$$

The two subclasses of minimal surfaces are characterized in terms of the invariants K and $\varkappa$ as follows:

- the class of minimal super-conformal surfaces is characterized by $K^2 - \varkappa^2 = 0$;
- the class of minimal surfaces of general type is characterized by $K^2 - \varkappa^2 > 0$.

The class of minimal super-conformal surfaces in $\mathbb{R}^4$ is locally equivalent to the class of holomorphic curves in $\mathbb{C}^2 \equiv \mathbb{R}^4$.

The surfaces with flat normal connection are characterized in terms of the ellipse of normal curvature as follows

Proposition 3.5. *Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. Then M^2 is a surface with flat normal connection if and only if for each point $p \in M^2$ the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.*

Proof: In [1] it is proved that the curvature of the normal connection $\varkappa$ of a surface M^2 in $\mathbb{R}^4$ is the Gauss torsion $\varkappa_G$ of M^2. The notion of the Gauss torsion is introduced by É. Cartan [2] for a p-dimensional submanifold of an n-dimensional Riemannian manifold and is given by the Euler curvatures. In case of a 2-dimensional surface M^2 in $\mathbb{R}^4$ the Gauss torsion at a point $p \in M^2$ is equal to $2ab$, where a and b are the semi-axis of the ellipse of normal curvature at p. Hence, $\varkappa = 0$ if and only if the ellipse of curvature is a line segment.

Let M^2 be a surface with flat normal connection, i.e. $\varkappa = 0$, $k \neq 0$. From (2.2) it follows, that $\nu_1 = \nu_2$. Further, equalities (2.1) imply that for each $v = \cos\psi\, x + \sin\psi\, y$, we have $\sigma(v,v) = H + \sin 2\psi(\lambda\, b + \mu\, l)$. So, when v goes once around the unit tangent circle, the vector $\sigma(v,v)$ goes twice along the line segment collinear with $\lambda\, b + \mu\, l$ and centered at H. The mean curvature vector field is $H = \nu_1\, b$. Since $k \neq 0$ then $\mu \neq 0$, and the line segment is not collinear with H. □

In case of $\lambda = 0$ the mean curvature vector field H is orthogonal to the line segment, while in case of $\lambda \neq 0$ the mean curvature vector field H is not orthogonal to the line segment. The length d of the line segment is

$$d = \sqrt{\lambda^2 + \mu^2} = \sqrt{H^2 - K}.$$

So, there arises a subclass of surfaces with flat normal connection, characterized by the conditions:

$$K = 0 \quad \text{or} \quad d = \|H\|.$$

Proposition 3.4 and Proposition 3.5 give us the following

Corollary 3.6. *Let M^2 be a surface in $\mathbb{R}^4$ free of flat points. Then the tangent indicatrix χ is a rectangular hyperbola (a Lorentz circle) if and only if the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.*

4. Examples of surfaces with flat normal connection

In this section we construct a family of surfaces with flat normal connection lying on a standard rotational hypersurface in $\mathbb{R}^4$.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{R}^4$, and $S^2(1)$ be a 2-dimensional sphere in $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$, centered at the origin O. We consider a smooth curve $c : l = l(v),\ v \in J,\ J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length ($l'^2(v) = 1$). We denote $t = l'$ and consider the moving frame field $\operatorname{span}\{t(v), n(v), l(v)\}$ of the curve c on $S^2(1)$. With respect to this orthonormal frame field the following Frenet formulas hold good:

$$\begin{aligned} l' &= t;\\ t' &= \kappa\, n - l;\\ n' &= -\kappa\, t, \end{aligned} \tag{4.1}$$

where κ is the spherical curvature of c.

Let $f = f(u),\ g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) + \dot{g}^2(u) = 1,\ u \in I$. Now we construct a surface M^2 in $\mathbb{R}^4$ in the following way:

$$M^2 : z(u,v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\ v \in J. \tag{4.2}$$

The surface M^2 lies on the rotational hypersurface M^3 in $\mathbb{R}^4$ obtained by the rotation of the meridian curve $m : u \rightarrow (f(u), g(u))$ around the Oe_4-axis in $\mathbb{R}^4$. Since M^2 consists of meridians of M^3, we call M^2 a *meridian surface.*

The tangent space of M^2 is spanned by the vector fields:

$$\begin{aligned} z_u &= \dot{f}\, l + \dot{g}\, e_4;\\ z_v &= f\, t, \end{aligned}$$

and hence the coefficients of the first fundamental form of M^2 are $E = 1;\ F = 0;\ G = f^2(u)$. Taking into account (4.1), we calculate the second partial derivatives of $z(u, v)$:

$$\begin{aligned} z_{uu} &= \ddot{f}\, l + \ddot{g}\, e_4;\\ z_{uv} &= \dot{f}\, t;\\ z_{vv} &= f\kappa\, n - f\, l. \end{aligned}$$

Let us denote $x = z_u,\ y = \dfrac{z_v}{f} = t$ and consider the following orthonormal normal frame field of M^2:

$$n_1 = n(v); \qquad n_2 = -\dot{g}(u)\, l(v) + \dot{f}(u)\, e_4.$$

Thus we obtain a positive orthonormal frame field $\{x, y, n_1, n_2\}$ of M^2. If we denote by κ_m the curvature of the meridian curve m, i.e. $\kappa_m(u) = \dot{f}(u)\ddot{g}(u) - \dot{g}(u)\ddot{f}(u) = \dfrac{-\ddot{f}(u)}{\sqrt{1 - \dot{f}^2(u)}}$,

then we get the following derivative formulas of M^2:

$$\begin{array}{llll}
 & \nabla'_x x = \kappa_m\, n_2; & & \nabla'_x n_1 = 0; \\
 & \nabla'_x y = 0; & & \nabla'_y n_1 = -\dfrac{\kappa}{f}\, y; \\
(4.3) & \nabla'_y x = \dfrac{\dot f}{f}\, y; & & \nabla'_x n_2 = -\kappa_m\, x; \\
 & \nabla'_y y = -\dfrac{\dot f}{f}\, x + \dfrac{\kappa}{f}\, n_1 + \dfrac{\dot g}{f}\, n_2; & & \nabla'_y n_2 = -\dfrac{\dot g}{f}\, y.
\end{array}$$

The coefficients of the second fundamental form of M^2 are $L = N = 0$, $M = -\kappa_m(u)\,\kappa(v)$. Taking into account (4.3), we find the invariants k, $\varkappa$, K:

$$k = -\frac{\kappa_m^2(u)\,\kappa^2(v)}{f^2(u)}; \qquad \varkappa = 0; \qquad K = \frac{\kappa_m(u)\,\dot g(u)}{f(u)}. \tag{4.4}$$

The equality $\varkappa = 0$ implies that M^2 is a surface with flat normal connection.

The mean curvature vector field H is given by

$$H = \frac{\kappa}{2f}\, n_1 + \frac{\dot g + f\kappa_m}{2f}\, n_2. \tag{4.5}$$

There are three main classes of meridian surfaces:

I. $\kappa = 0$, i.e. the curve c is a great circle on $S^2(1)$. In this case $n_1 = const$, and M^2 is a planar surface lying in the constant 3-dimensional space spanned by $\{x, y, n_2\}$. Particularly, if in addition $\kappa_m = 0$, i.e. the meridian curve lies on a straight line, then M^2 is a developable surface in the 3-dimensional space span$\{x, y, n_2\}$.

II. $\kappa_m = 0$, i.e. the meridian curve is part of a straight line. In such case $k = \varkappa = K = 0$, and M^2 is a developable ruled surface. If in addition $\kappa = const$, i.e. c is a circle on $S^2(1)$, then M^2 is a developable ruled surface in a 3-dimensional space. If $\kappa \neq const$, i.e. c is not a circle on $S^2(1)$, then M^2 is a developable ruled surface in $\mathbb{R}^4$.

III. $\kappa_m\,\kappa \neq 0$, i.e. c is not a great circle on $S^2(1)$, and m is not a straight line. In this general case the invariant function $k < 0$, which implies that there exist two systems of asymptotic lines on M^2. The parametric lines of M^2 given by (4.2) are orthogonal and asymptotic.

Let M^2 be a meridian surface of the general class. Now we are going to find the meridian surfaces with:

- constant Gauss curvature K;
- constant mean curvature;
- constant invariant function k.

Proposition 4.1. *Let M^2 be a meridian surface in $\mathbb{R}^4$. Then M^2 has constant non-zero Gauss curvature K if and only if the meridian m is given by*

$$\begin{array}{ll}
f(u) = \alpha\cos\sqrt{K}u + \beta\sin\sqrt{K}u, & K > 0; \\
f(u) = \alpha\cosh\sqrt{-K}u + \beta\sinh\sqrt{-K}u, & K < 0,
\end{array}$$

where α and β are constants.

Proof: Using (4.4) and $\dot f^2 + \dot g^2 = 1$, we obtain that M^2 has constant Gauss curvature $K \neq 0$ if and only if the meridian m satisfies the following differential equation

$$\ddot f(u) + K f(u) = 0.$$

The general solution of the above equation is given by

$$\begin{array}{ll} f(u) = \alpha \cos \sqrt{K} u + \beta \sin \sqrt{K} u, & \text{in case} \quad K > 0; \\ f(u) = \alpha \cosh \sqrt{-K} u + \beta \sinh \sqrt{-K} u, & \text{in case} \quad K < 0, \end{array}$$

where α and β are constants. The function $g(u)$ is determined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$. □

The equality (4.5) implies that the mean curvature of M^2 is given by

$$||H|| = \sqrt{\frac{\kappa^2(v) + \left(\dot{g}(u) + f(u)\kappa_m(u)\right)^2}{4f^2(u)}}. \tag{4.6}$$

The meridian surfaces with constant mean curvature (CMC meridian surfaces) are described in

Proposition 4.2. *Let M^2 be a meridian surface in $\mathbb{R}^4$. Then M^2 has constant mean curvature $||H|| = a = const$, $a \neq 0$ if and only if the curve c on $S^2(1)$ is a circle with constant spherical curvature $\kappa = const = b$, $b \neq 0$, and the meridian m is determined by the following differential equation:*

$$\left(1 - \dot{f}^2 - f\ddot{f}\right)^2 = (1 - \dot{f}^2)(4a^2 f^2 - b^2).$$

Proof: From (4.6) it follows that $||H|| = a$ if and only if

$$\kappa^2(v) = 4a^2 f^2(u) - (\dot{g}(u) + f(u)\kappa_m(u))^2,$$

which implies

$$\begin{aligned} &\kappa = const = b, \; b \neq 0; \\ &4a^2 f^2(u) - (\dot{g}(u) + f(u)\kappa_m(u))^2 = b^2. \end{aligned} \tag{4.7}$$

The first equality of (4.7) implies that the spherical curve c has constant spherical curvature $\kappa = b$, i.e. c is a circle. Using that $\dot{f}^2 + \dot{g}^2 = 1$, and $\kappa_m = \dot{f}\ddot{g} - \dot{g}\ddot{f}$ we calculate that $\dot{g} + f\kappa_m = \dfrac{1 - \dot{f}^2 - f\ddot{f}}{\sqrt{1 - \dot{f}^2}}$. Hence, the second equality of (4.7) gives the following differential equation for the meridian m:

$$\left(1 - \dot{f}^2 - f\ddot{f}\right)^2 = (1 - \dot{f}^2)(4a^2 f^2 - b^2). \tag{4.8}$$

Further, if we set $\dot{f} = y(f)$ in equation (4.8), we obtain that the function $y = y(t)$ is a solution of the following differential equation

$$1 - y^2 - \frac{t}{2}(y^2)' = \sqrt{1 - y^2}\sqrt{4a^2t^2 - b^2}.$$

The general solution of the above equation is given by

$$y(t) = \sqrt{1 - \frac{1}{t^2}\left(C + \frac{t}{2}\sqrt{4a^2t^2 - b^2} - \frac{b^2}{4a}\ln|2at + \sqrt{4a^2t^2 - b^2}|\right)^2}; \qquad C = const. \tag{4.9}$$

The function $f(u)$ is determined by $\dot{f} = y(f)$ and (4.9). The function $g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$. □

At the end of this section we shall find the meridian surfaces with constant invariant k.

Proposition 4.3. *Let M^2 be a meridian surface in $\mathbb{R}^4$. Then M^2 has a constant invariant $k = const = -a^2$, $a \neq 0$ if and only if the curve c on $S^2(1)$ is a circle with spherical curvature $\varkappa = const = b$, $b \neq 0$, and the meridian m is determined by the following differential equation:*

$$\ddot{f}(u) = \mp \frac{a}{b}\, f(u)\sqrt{1 - \dot{f}^2(u)}.$$

Proof: Using (4.4) we obtain that $k = const = -a^2$, $a \neq 0$ if and only if $\varkappa^2(v)\varkappa_m^2(u) = a^2 f^2(u)$. Hence,

$$\varkappa(v) = \pm\, a\, \frac{f(u)}{\varkappa_m(u)}.$$

The last equality implies

$$\begin{aligned} &\varkappa = const = b, \; b \neq 0; \\ &\pm\, a\, \frac{f(u)}{\varkappa_m(u)} = b. \end{aligned} \tag{4.10}$$

The first equality of (4.10) implies that the spherical curve c has constant spherical curvature $\varkappa = b$, i.e. c is a circle. The second equality of (4.10) gives the following differential equation for the function $f(u)$:

$$\frac{\ddot{f}(u)}{\sqrt{1 - \dot{f}^2(u)}} = \mp \frac{a}{b}\, f(u). \tag{4.11}$$

Again setting $\dot{f} = y(f)$ in equation (4.11), we obtain that the function $y = y(t)$ is a solution of the following differential equation

$$\frac{y y'}{\sqrt{1 - y^2}} = \mp \frac{a}{b}\, t.$$

The general solution of the above equation is given by

$$y(t) = \sqrt{1 - \left(C \pm \frac{a}{b}\,\frac{t^2}{2}\right)^2}; \qquad C = const. \tag{4.12}$$

The function $f(u)$ is determined by $\dot{f} = y(f)$ and (4.12). The function $g(u)$ is defined by $\dot{g}(u) = \sqrt{1 - \dot{f}^2(u)}$. □

5. Examples of surfaces consisting of parabolic points

In this section we shall find the generalized (in the sense of C. Moore) rotational surfaces in $\mathbb{R}^4$, consisting of parabolic points.

We consider a surface M^2 in $\mathbb{R}^4$ given by

$$z(u,v) = (f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\cos\beta v, g(u)\sin\beta v)\,; \quad u \in J \subset \mathbb{R},\; v \in [0; 2\pi), \tag{5.1}$$

where $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2(u) + \beta^2 g^2(u) > 0$, $f'^{\,2}(u) + g'^{\,2}(u) > 0$, $u \in J$, and α, β are positive constants.

Each parametric curve $u = u_0 = const$ of M^2 is given by

$$c_v : z(v) = (a\cos\alpha v, a\sin\alpha v, b\cos\beta v, b\sin\beta v)\,; \quad a = f(u_0),\; b = g(u_0)$$

and its Frenet curvatures are

$$\varkappa_{c_v} = \sqrt{\frac{a^2\alpha^4 + b^2\beta^4}{a^2\alpha^2 + b^2\beta^2}}; \quad \tau_{c_v} = \frac{ab\alpha\beta(\alpha^2 - \beta^2)}{\sqrt{a^2\alpha^4 + b^2\beta^4}\sqrt{a^2\alpha^2 + b^2\beta^2}}; \quad \sigma_{c_v} = \frac{\alpha\beta\sqrt{a^2\alpha^2 + b^2\beta^2}}{\sqrt{a^2\alpha^4 + b^2\beta^4}}.$$

Hence, in case of $\alpha \neq \beta$ each parametric curve $u = const$ is a curve in $\mathbb{R}^4$ with constant curvatures, and in case of $\alpha = \beta$ each parametric curve $u = const$ is a circle.

Each parametric curve $v = v_0 = const$ of M^2 is given by

$$c_u : z(u) = (\, A_1 f(u), A_2 f(u), B_1 g(u), B_2 g(u)\,)\,,$$

where $A_1 = \cos\alpha v_0$, $A_2 = \sin\alpha v_0$, $B_1 = \cos\beta v_0$, $B_2 = \sin\beta v_0$. The Frenet curvatures of c_u are expressed as follows:

$$\varkappa_{c_u} = \frac{|g' f'' - f' g''|}{(\sqrt{f'^{\,2} + g'^{\,2}})^3}; \quad \tau_{c_u} = 0.$$

Hence, c_u is a plane curve with curvature $\varkappa_{c_u} = \dfrac{|g' f'' - f' g''|}{(\sqrt{f'^{\,2} + g'^{\,2}})^3}$. So, for each $v = const$ the parametric curves c_u are congruent in $\mathbb{R}^4$. We call these curves *meridians* of M^2.

Considering general rotations in $\mathbb{R}^4$, C. Moore introduced general rotational surfaces [6] (see also [7, 8]). The surface M^2, given by (5.1) is a general rotational surface whose meridians lie in two-dimensional planes.

The tangent space of M^2 is spanned by the vector fields

$$\begin{aligned} z_u &= (f' \cos\alpha v, f' \sin\alpha v, g' \cos\beta v, g' \sin\beta v)\,; \\ z_v &= (-\alpha f \sin\alpha v, \alpha f \cos\alpha v, -\beta g \sin\beta v, \beta g \cos\beta v)\,. \end{aligned}$$

Hence, the coefficients of the first fundamental form are $E = f'^{\,2}(u) + g'^{\,2}(u)$; $F = 0$; $G = \alpha^2 f^2(u) + \beta^2 g^2(u)$ and $W = \sqrt{(f'^{\,2} + g'^{\,2})(\alpha^2 f^2 + \beta^2 g^2)}$. We consider the following orthonormal tangent frame field

$$\begin{aligned} x &= \frac{1}{\sqrt{f'^{\,2} + g'^{\,2}}}\,(f' \cos\alpha v, f' \sin\alpha v, g' \cos\beta v, g' \sin\beta v); \\ y &= \frac{1}{\sqrt{\alpha^2 f^2 + \beta^2 g^2}}\,(-\alpha f \sin\alpha v, \alpha f \cos\alpha v, -\beta g \sin\beta v, \beta g \cos\beta v). \end{aligned}$$

The second partial derivatives of $z(u, v)$ are expressed as follows

$$\begin{aligned} z_{uu} &= (f'' \cos\alpha v, f'' \sin\alpha v, g'' \cos\beta v, g'' \sin\beta v)\,; \\ z_{uv} &= (-\alpha f' \sin\alpha v, \alpha f' \cos\alpha v, -\beta g' \sin\beta v, \beta g' \cos\beta v)\,; \\ z_{vv} &= (-\alpha^2 f \cos\alpha v, -\alpha^2 f \sin\alpha v, -\beta^2 g \cos\beta v, -\beta^2 g \sin\beta v)\,. \end{aligned}$$

Now let us consider the following orthonormal normal frame field

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{f'^{\,2} + g'^{\,2}}}\,(g' \cos\alpha v, g' \sin\alpha v, -f' \cos\beta v, -f' \sin\beta v); \\ n_2 &= \frac{1}{\sqrt{\alpha^2 f^2 + \beta^2 g^2}}\,(-\beta g \sin\alpha v, \beta g \cos\alpha v, \alpha f \sin\beta v, -\alpha f \cos\beta v). \end{aligned}$$

It is easy to verify that $\{x, y, n_1, n_2\}$ is a positive oriented orthonormal frame field in $\mathbb{R}^4$.

We calculate the functions c_{ij}^k, $i, j, k = 1, 2$:

$$c_{11}^1 = g(z_{uu}, n_1) = \frac{g'f'' - f'g''}{\sqrt{f'^{\,2} + g'^{\,2}}}; \qquad c_{11}^2 = g(z_{uu}, n_2) = 0;$$

$$c_{12}^1 = g(z_{uv}, n_1) = 0; \qquad c_{12}^2 = g(z_{uv}, n_2) = \ ds\frac{\alpha\beta(gf' - fg')}{\sqrt{\alpha^2 f^2 + \beta^2 g^2}};$$

$$c_{22}^1 = g(z_{vv}, n_1) = \frac{\beta^2 g f' - \alpha^2 f g'}{\sqrt{f'^{\,2} + g'^{\,2}}}; \qquad c_{22}^2 = g(z_{vv}, n_2) = 0.$$

Therefore the coefficients L, M and N of the second fundamental form of M^2 are expressed as follows:

$$L = \frac{2\alpha\beta(gf' - fg')(g'f'' - f'g'')}{(\alpha^2 f^2 + \beta^2 g^2)(f'^{\,2} + g'^{\,2})}; \qquad M = 0; \qquad N = \frac{-2\alpha\beta(gf' - fg')(\beta^2 g f' - \alpha^2 f g')}{(\alpha^2 f^2 + \beta^2 g^2)(f'^{\,2} + g'^{\,2})}.$$

Consequently, the invariants k, $\varkappa$ and K of M^2 are:

$$k = \frac{-4\alpha^2\beta^2(gf' - fg')^2(g'f'' - f'g'')(\beta^2 g f' - \alpha^2 f g')}{(\alpha^2 f^2 + \beta^2 g^2)^3(f'^{\,2} + g'^{\,2})^3};$$

$$\varkappa = \frac{\alpha\beta(gf' - fg')}{(\alpha^2 f^2 + \beta^2 g^2)^2(f'^{\,2} + g'^{\,2})^2}\left((\alpha^2 f^2 + \beta^2 g^2)(g'f'' - f'g'') - (f'^{\,2} + g'^{\,2})(\beta^2 g f' - \alpha^2 f g')\right);$$

$$K = \frac{(\alpha^2 f^2 + \beta^2 g^2)(\beta^2 g f' - \alpha^2 f g')(g'f'' - f'g'') - \alpha^2\beta^2(f'^{\,2} + g'^{\,2})(gf' - fg')^2}{(\alpha^2 f^2 + \beta^2 g^2)^2(f'^{\,2} + g'^{\,2})^2}.$$

Now we shall find the generalized rotational surfaces with $k = 0$. Without loss of generality we assume that the meridian m is defined by $f = u$; $g = g(u)$. Then

$$k = \frac{4\alpha^2\beta^2(g - ug')^2 g''(\beta^2 g - \alpha^2 u g')}{(\alpha^2 u^2 + \beta^2 g^2)^3(1 + g'^{\,2})^3};$$

The invariant k is zero in the following three cases:

1. $g(u) = a\,u$, $a = const \neq 0$. In that case $k = \varkappa = K = 0$, and M^2 is a developable surface in $\mathbb{R}^4$.

2. $g(u) = a\,u + b$, $a = const \neq 0, b = const \neq 0$. In this case $k = 0$, but $\varkappa \neq 0$, $K \neq 0$. Consequently, M^2 is a non-developable ruled surface in $\mathbb{R}^4$.

3. $g(u) = c\,u^{\frac{\beta^2}{\alpha^2}}$, $c = const \neq 0$. In case of $\alpha \neq \beta$ we get $k = 0$, and the invariants $\varkappa$ and K are given by

$$\varkappa = \frac{c^2\beta^3(\beta^2 - \alpha^2)^2 u^{2\frac{\beta^2 - \alpha^2}{\alpha^2}}}{\alpha^5\left(\alpha^2 u^2 + \beta^2 c^2 u^{2\frac{\beta^2}{\alpha^2}}\right)\left(1 + c^2\frac{\beta^4}{\alpha^4} u^{2\frac{\beta^2 - \alpha^2}{\alpha^2}}\right)^2};$$

$$K = -\frac{c^2\beta^2(\beta^2 - \alpha^2)^2 u^{2\frac{\beta^2}{\alpha^2}}}{\alpha^2\left(\alpha^2 u^2 + \beta^2 c^2 u^{2\frac{\beta^2}{\alpha^2}}\right)^2\left(1 + c^2\frac{\beta^4}{\alpha^4} u^{2\frac{\beta^2 - \alpha^2}{\alpha^2}}\right)}.$$

Hence, $\varkappa \neq 0$, $K \neq 0$. In this case the parametric lines $u = const$ and $v = const$ are not straight lines. This is a non-trivial example of generalized rotational surfaces with $k = 0$.

Acknowledgements: The second author is partially supported by "L. Karavelov" Civil Engineering Higher School, Sofia, Bulgaria under Contract No 10/2009.

References

[1] Aminov Yu. *The geometry of submanifolds.* Gordon and Breach Science Publishers, 2002.

[2] Cartan É. *Riemannian geometry in an orthogonal frame. From lectures delivered by Élie Cartan at the Sorbonne 1926–27.* Singapore: World Scientific, 2001.

[3] Dajczer, M. and R. Tojeiro. *All superconformal surfaces in $\mathbb{R}^4$ in terms of minimal surfaces.* Math. Z. **261** (2009), 4, 869-890.

[4] Ganchev G. and V. Milousheva. *On the theory of surfaces in the four-dimensional Euclidean space.* Kodai Math. J., **31** (2008), 183-198.

[5] Ganchev G. and V. Milousheva. *Minimal surfaces in the four-dimensional Euclidean space.* arXiv:0806.3334v1

[6] Moore C. *Surfaces of rotation in a space of four dimensions*. The Annals of Math., 2nd Ser., **21** (1919), 2, 81-93.

[7] Moore C. and E. Wilson. *A general theory of surfaces*. J. Nat. Acad. Proc. **2** (1916), 273-278.

[8] Moore C. and E. Wilson. *Differential geometry of two-dimensional surfaces in hyperspaces*. Proc. Acad. Arts Sci. **52** (1916), 267-368.

Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, Acad. G. Bonchev Str. bl. 8, 1113 Sofia, Bulgaria

E-mail address: `ganchev@math.bas.bg`

Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, Acad. G. Bonchev Str. bl. 8, 1113, Sofia, Bulgaria

E-mail address: `vmil@math.bas.bg`